Subregular Affine Kazhdan-Lusztig Polynomials in Type D

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PRIMES October Conference

October 13, 2024

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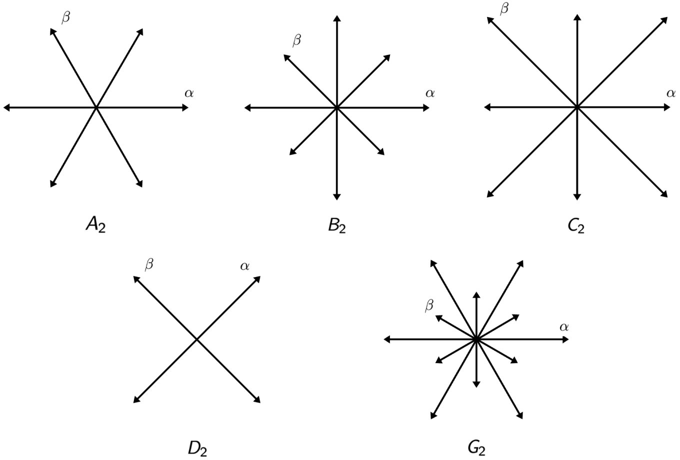
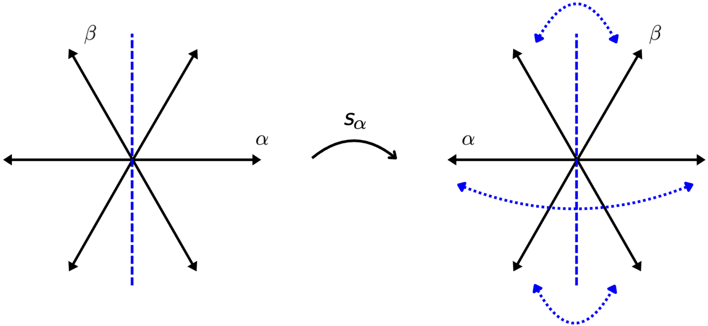
Root Systems

- Root systems are certain symmetric arrangement of vectors.

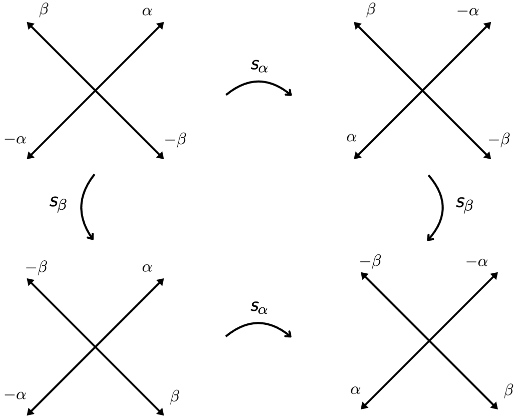


Weyl Groups

- A Weyl group is a group of symmetries of root systems generated by reflections across hyperplanes orthogonal to the roots.
- Let s_α denote the reflection across the hyperplane orthogonal to the root α.

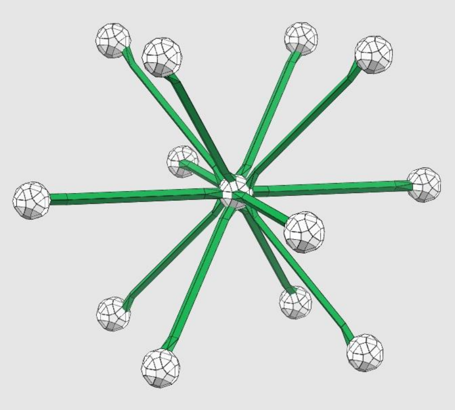


Weyl Group of type D_2



D_2 has order 4, consisting of the four symmetries above.

Root System of type D_3



- D_3 has 12 roots, and the Weyl group has order 24.
- The root system can be seen as connecting the center of a cube with the midpoint of the cube's edges.

Root System of type D_n

- The root system of D_n can be expressed as

$$\Phi = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\},$$

where ϵ_i are the unit vectors of the vector space.

- The simple roots of D_n are

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}.$$

because every root of D_n is a $\mathbb{Z}$-linear combination of roots from this set.

Weyl Group of type D_n

- The reflection across the hyperplane orthogonal to the simple root $\epsilon_i - \epsilon_{i+1}$ sends

$$(a_1, a_2, \ldots, a_n) \mapsto (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n),$$

while the simple root $\epsilon_{n-1} + \epsilon_n$ sends

$$(a_1, a_2, \ldots, a_n) \mapsto (a_1, a_2, \ldots, -a_n, -a_{n-1}).$$

- More generally, $\epsilon_i - \epsilon_j$ swaps a_i and a_j, and $\epsilon_i + \epsilon_j$ swaps a_i with $-a_j$ and a_j with $-a_i$.

Signed permutations

- Other than the description through reflections, Weyl groups of type D_n can be expressed through even signed permutations of $\{1, \dots, n\}$.
- A signed permutation looks like

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \overline{1} & \overline{3} \end{pmatrix}$$

They compose like:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \overline{1} & \overline{3} \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \overline{1} & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \overline{2} & \overline{3} & \overline{1} \end{pmatrix}.$$

- A signed permutation is even if the number of bars is even.

Simple Reflections

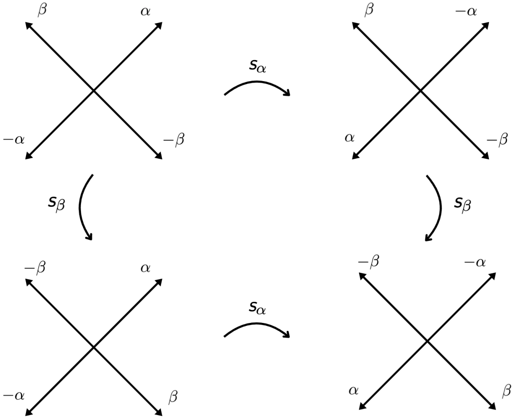
If α is a simple root, s_α is called a simple reflection.

Theorem

Let $\{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots of Weyl group W. Then the set of simple reflections $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ generates W.

What are the relations between these generators?

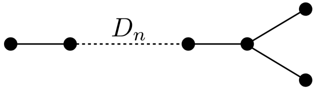
Example: D2



Here, s_α and s_β are the simple reflections.

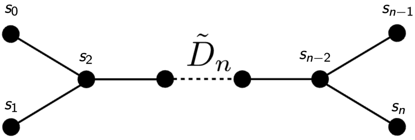
Dynkin Diagrams

This is the Dynkin diagram of Type D_n.



- Each vertex corresponds to a simple root of D_n.
- A pair of simple roots α_i and α_j are not connected by an edge when they are orthogonal to each other. As we saw in type D_2, this means $s_i s_j = s_j s_i$.
- Simple roots α_i and α_j are connected by an edge when they have a 120° angle between them. Then, $(s_i s_j)^3 = 1$ because $s_i s_j$ is a 240° rotation.
- This gives you all the relations!

Weyl Group of Type $\widetilde{D}_n$



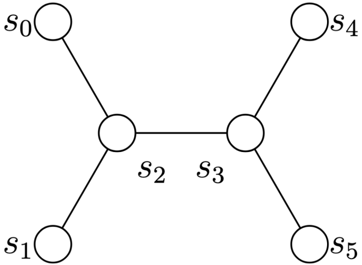
- We can define $\widetilde{D}_n$ with the following Dynkin diagram that has one more simple root.
- The order of the Weyl group is infinite!
- These Weyl groups are called affine Weyl groups.

Subregular Cell

Let c_{subreg} denote the set of all non-identity elements in W with a unique reduced word. Then c_{subreg} is called the subregular cell of W.

Example:

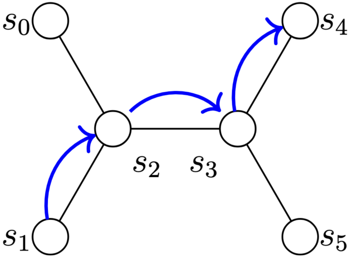
In $\widetilde{D}_5$, the element $s_1 s_2$ is in the subregular cell because it is unique, while the element $s_1 s_2 s_1$ is not in the subregular cell because $(s_1 s_2)^3 = 1$ so $s_1 s_2 s_1 = s_2 s_1 s_2$.



Subregular Cell of type $\widetilde{D}_n$

Theorem (Vasily Krylov, Kenta Suzuki, 2024)

In Weyl group W of type $\widetilde{D}_n$, if $w = s_{i_1} \cdots s_{i_n}$ is in the subregular cell, then w defines a path $(i_1, \ldots, i_n)$ on the Dynkin diagram such that each edge only appears once.



Choosing the starting and ending points of a path on the Dynkin diagram of $\widetilde{D}_n$ uniquely determines a subregular element.

Kazhdan-Lusztig Polynomials

- Kazhdan-Lusztig polynomials $\mathbf{m}_v^w$ are certain polynomials attached to Weyl groups with connections to representation theory and physics.
- Subregular Kazhdan-Lusztig polynomials are Kazhdan-Lusztig polynomials when v is subregular.
- Bezrukavnikov, Kac, and Krylov computed $\mathbf{m}_v^w$ when v is subregular and it ends with s_0 of Weyl groups in types $\widetilde{A}$, $\widetilde{D}$, and $\widetilde{E}$.

Theorem (K.)

Explicit formulas for $\mathbf{m}_v^w$ for all v subregular, in types $\widetilde{D}_4$ and $\widetilde{D}_5$.

Kazhdan-Lusztig polynomials of $\widetilde{D}_4$

Let $\gamma = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4$ where α_i is the corresponding root to the simple reflection s_i.

Let w_i denote the subregular element defined by the path on the Dynkin diagram starting at s_2 and ending at s_i.

Computing with the matrices, we find the following results:

$$\begin{aligned}
\mathbf{m}_{w_0}^{w_\gamma} &= (\gamma,\gamma) - (\gamma,\alpha_0) \\
\mathbf{m}_{w_1}^{w_\gamma} &= (\gamma,\gamma) - (\gamma,\alpha_1) \\
\mathbf{m}_{w_2}^{w_\gamma} &= 2(\gamma,\gamma) \\
\mathbf{m}_{w_3}^{w_\gamma} &= (\gamma,\gamma) - (\gamma,\alpha_3) \\
\mathbf{m}_{w_4}^{w_\gamma} &= (\gamma,\gamma) - (\gamma,\alpha_4)
\end{aligned}$$

Computations in $\widetilde{D}_5$

We use the same method for computations in $\widetilde{D}_5$ to find the following results:

$$\mathbf{m}_{w_0}^{w_\gamma} = (\gamma, \gamma) - (\gamma, \alpha_0)$$
$$\mathbf{m}_{w_1}^{w_\gamma} = (\gamma, \gamma) - (\gamma, \alpha_1)$$
$$\mathbf{m}_{w_2}^{w_\gamma} = 2(\gamma, \gamma)$$
$$\mathbf{m}_{w_3}^{w_\gamma} = 2(\gamma, \gamma) - (\gamma, \epsilon_3)$$
$$\mathbf{m}_{w_4}^{w_\gamma} = (\gamma, \gamma) - (\gamma, \alpha_4 + \epsilon_3)$$
$$\mathbf{m}_{w_4}^{w_\gamma} = (\gamma, \gamma) - (\gamma, \alpha_5 + \epsilon_3)$$

Acknowledgements

I would like to thank my mentor, Kenta, for proposing this project and for guidance during the preparation of this project.

I would also like to thank the MIT PRIMES program for arranging such an engaging research experience.

References

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